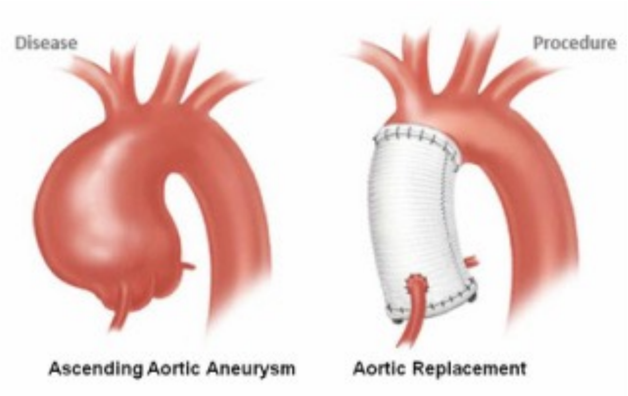
Georg Fantner

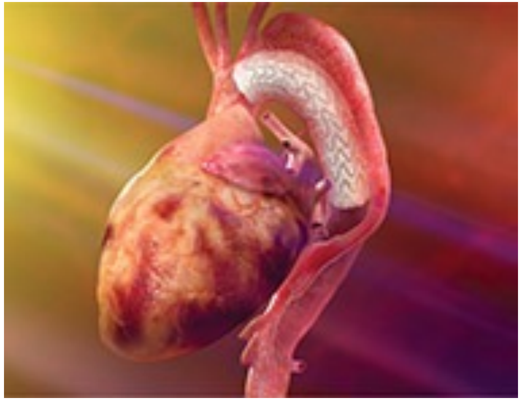
Thin-walled pressure vessel

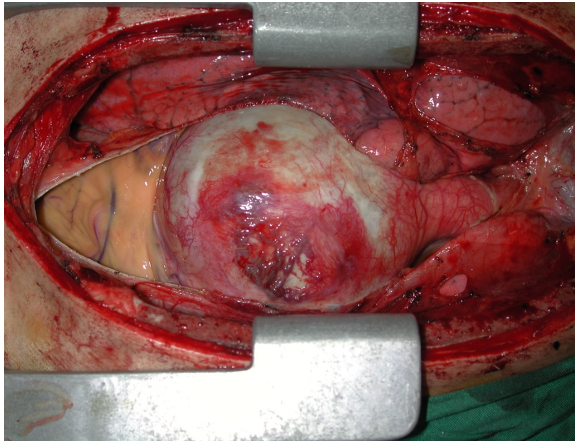
Why a hotdog always ruptures along its length…

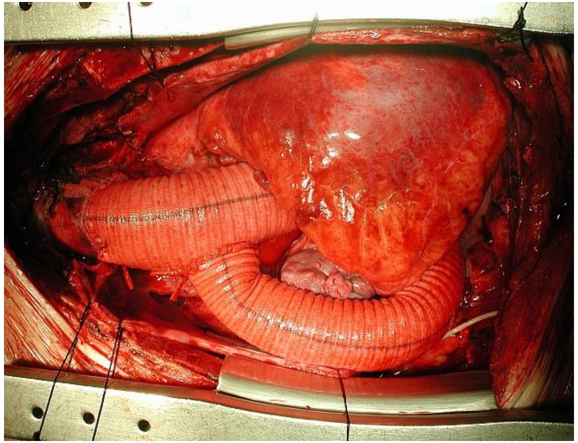
Thin walled pressure vessels

Aortic Aneurysm

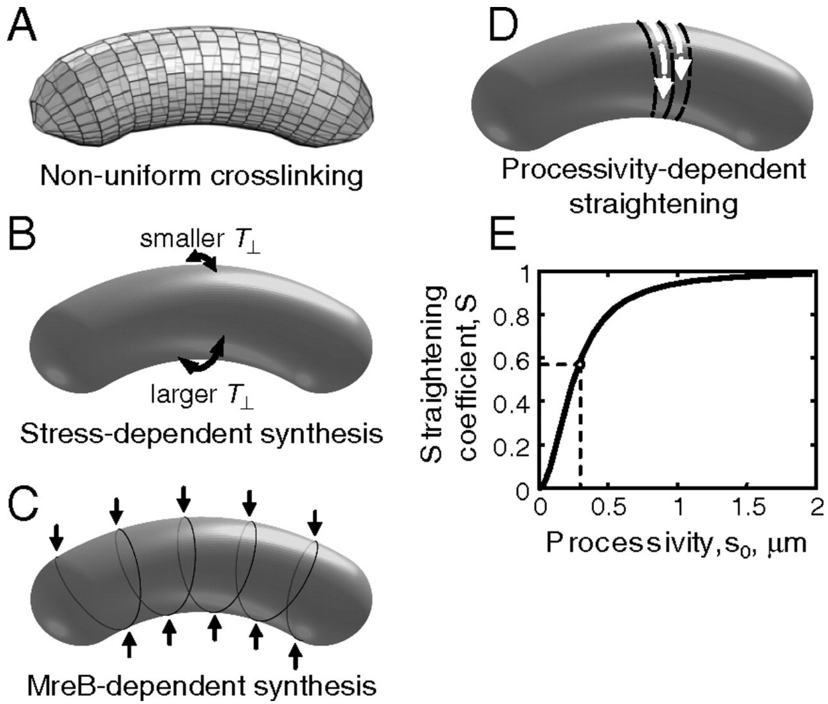


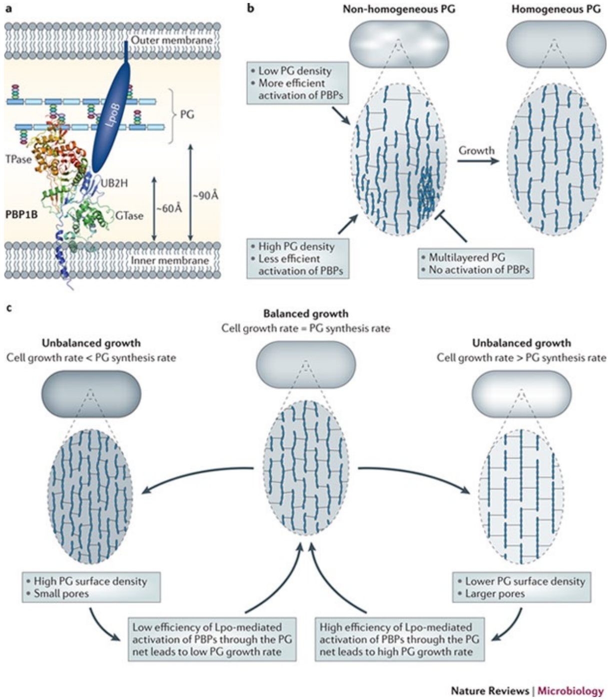






Bacteria as pressure vessels: The effect of Turgor pressure





Thin-walled pressure vessels

- Pressure vessels are generally combinations of spheres, cylinders or ellipsoids, with the task of containing gasses or liquids under pressure.
- We are interested in the stresses that occur in the walls of the pressure vessel.
- We call a pressure vessel *thin walled* if the thickness $t<0.1r_i$ inner radius (examples: boiler, scuba tank, inflated balloon). In this case the wall acts like a membrane and experiences no bending, no significant variation in the stress from the inner to the outer surface.
- We call a pressure vessel thick walled if $t>0.1r_i$ (examples: gun barrel, explosion chamber, high pressure hydraulic presses)

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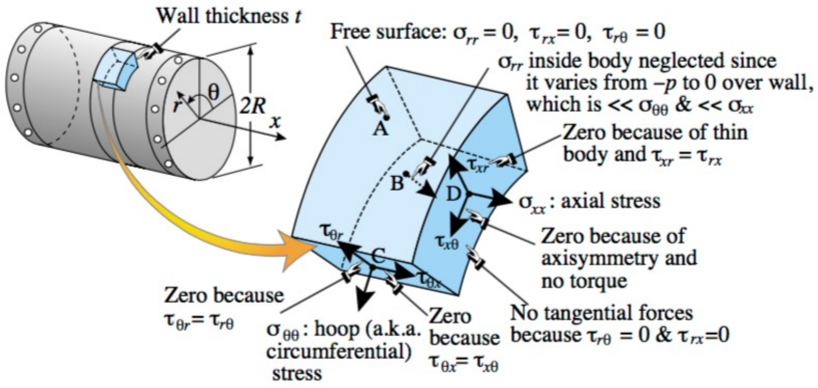


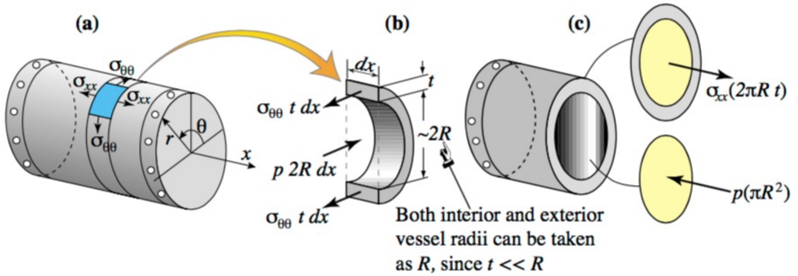
$$\begin{bmatrix} \sigma_{xx} & \tau_{x\theta} & \tau_{xr} \\ \tau_{\theta x} & \sigma_{\theta\theta} & \tau_{\theta r} \\ \tau_{rx} & \tau_{r\theta} & \sigma_{rr} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_{xx} = \text{Axial stress}$$

$$\sigma_{\theta\theta} = \text{Hoop stress}$$

Thin-walled pressure vessels

- We recall: pressure exerts a force/unit area, normal to the area (=parallel to the normal vector of the area)
- This pressure induces a tensile stress in the thin wall.
- The thin wall is in *plane stress!*
- Since the pressure does not apply any shear loads, the shear components on all sides have to be zero.
- In cylindrical coordinates



Thin walled pressure vessels

Look at the cross section from the side:

Projected “inner” area $D_i L$

Wall area: 2 t L

Hoop-stresses

- The tensile stresses around the circumference are called *hoop stresses*. In cylindrical coordinates they would be called:

$$\sigma_{\theta\theta} = \sigma_\theta := \sigma_1$$

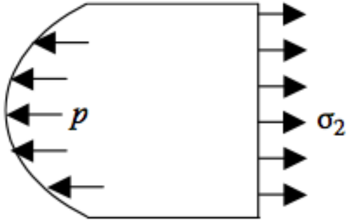
- From the equilibrium of forces in the “out of plane” direction we get:

$$p \cdot r_i \cdot dx = \sigma_{\theta\theta} \cdot 2t \cdot dx$$

- Hoop stress:

$$\boxed{\sigma_1 = \frac{p\, D_i}{2t} = \frac{p\, r_i}{t}}$$

Longitudinal stresses



- *Axial or longitudinal stresses* σ_2: stresses acting along the axis of the pressure vessel.
- The projected cross-section of the end caps is a circle with area:

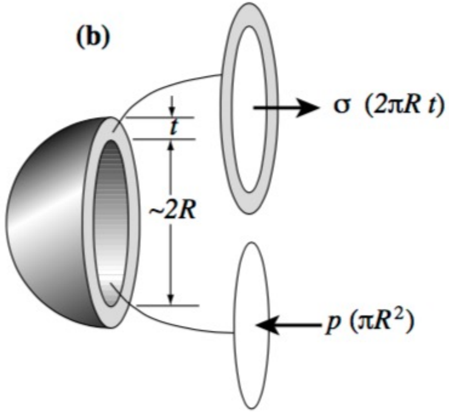
$$A = \pi r_i^2$$

- The force in the longitudinal direction is therefore:
- F_p is balanced by the longitudinal stress $\sigma_{xx}=\sigma_x=\sigma_2$ on the area of the cross-section

$$F_p = p \cdot A = p \cdot \pi\, r_i^2$$
$$F_\sigma = \sigma_x \cdot \pi(r_0^2 - r_i^2)$$

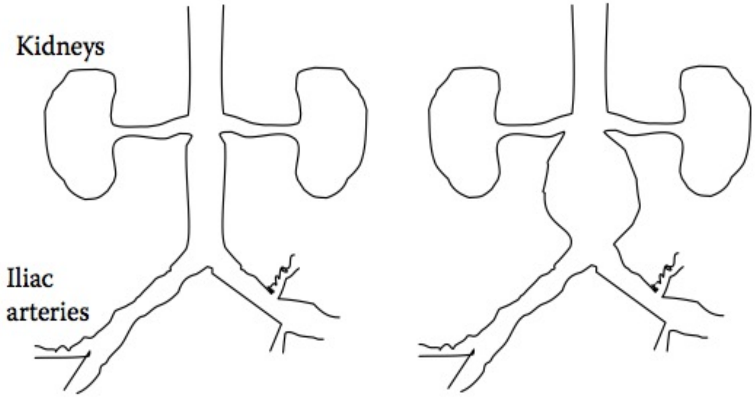
- From force equilibrium:

$$\boxed{\sigma_2 = \frac{pr}{2t}}$$



- For a thin walled <u>spherical</u> pressure vessel:

$$\sigma_1 = \sigma_2 = \frac{pr}{2t}$$



Example: arterial aneurism

Aneurism is a condition where there is ballooning or dilation in a blood vessel.

- Calculate the hoop stress in a healthy artery modeled as a cylinder (r_i=1cm)
- Calculate the hoop stress in a dilated artery modeled as a cylinder (r_i=2.5cm).
- Calculate the hoop stress in a ballooning artery with d=5cm

Assume: thin-walled pressure vessel, blood pressure varies from low(diabolic) to high (systolic) pressure during one heartbeat: $p_{systolic}$=1.6 N/cm^2. The artery has a wall thickness of 1mm.

Transformation of stresses and strains

Transformation of plane stress

Stress in axially loaded bars

What happens inside the bar when it is stretched?

- Axially loaded bar fixed at x=0 and loaded by force P at x=L (figure a)
- Apply method of section normal to bar axis (figure b)

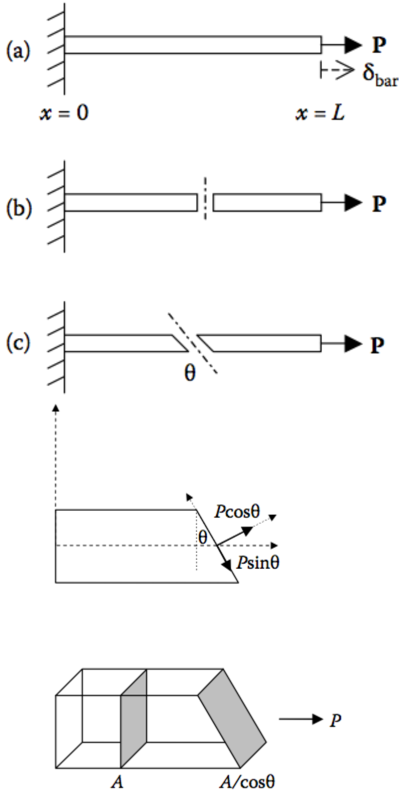
$$\sigma \equiv \frac{P}{A}$$

- Apply method of section angled to bar axis (figure c)

Angular dependence of stress

$$\sigma_\theta = \frac{force}{area} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta$$

$$\tau_\theta = -\frac{P\sin\theta}{A/\cos\theta} = -\frac{P}{A}\sin\theta\cos\theta$$



Stress in axially loaded bars

Directions of maximum stress

- Maximum normal stress:

$$\sigma_\theta = \frac{P}{A}\cos^2\theta \rightarrow \boxed{\sigma_{max} = \frac{P}{A}} @\, \theta = 0$$

- Maximum shear stress:

$$\frac{d}{d\theta}\tau_\theta = \frac{d}{d\theta}\left(-\frac{P}{A}\sin\theta\cos\theta\right) = 0 \rightarrow \theta = 45^o$$

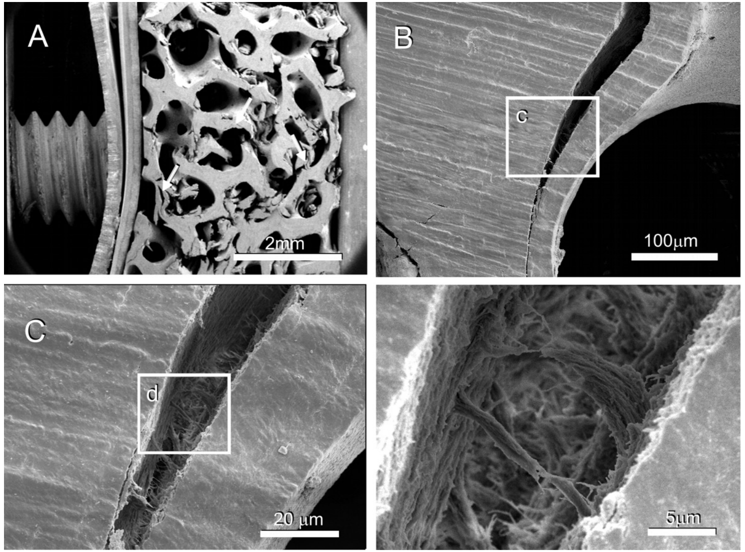
$$|\tau_{max}| = \frac{P}{A}\sin(\pi/4)\cos(\pi/4)$$

$$\boxed{|\tau_{max}| = \frac{P}{2A} = \frac{\sigma_{max}}{2}}$$

EPFL

Transformation of stress and strain

- So far, we’ve looked at isolated effects of normal and shear stress due to loading by axial forces and shear forces
- We’ve always looked at what the effects of stresses are along the direction that they are acting on (or in the direction 90degr. from that) to see if we are within the safe stress limits of the material
- However, a combination of normal stress and shear stress can result in much larger normal stresses in a different direction.
- To calculate these maximum stress values and the angles in which they occur, we need to find a way to calculate the stress in any direction that is oriented at an arbitrary angle to our reference axes



Fantner et al. ***Bone*** *2004*

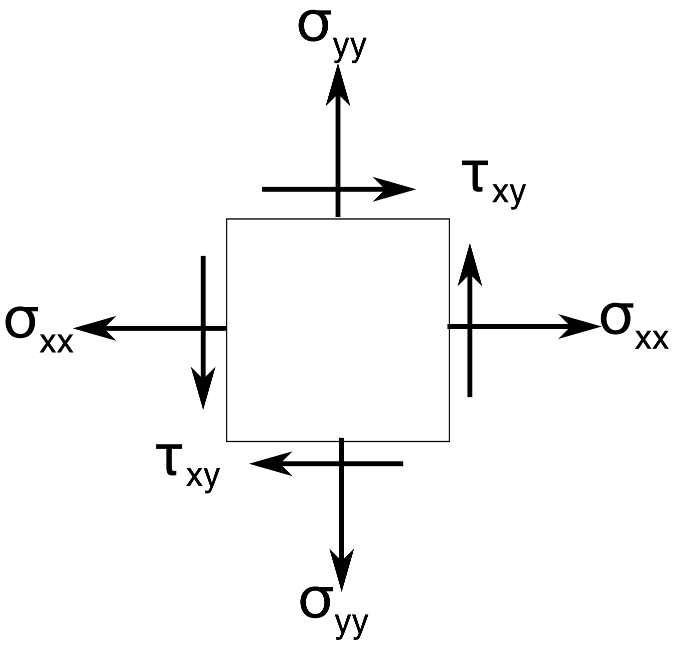
- Bone consists of mineralized collagen fibrils.
- Bone fracture occurs by delamination along the fibrils

Why do we care about stress transformations?

Many materials are not uniformly strong in all directions!

Very important for composite materials

Carbon composite is very strong in some direction, much weaker in other directions.



Transformation of stress and strain

What if we have multiple loads acting at the same time?

A square sheet of material is loaded in the X and Y direction

Let us assume:

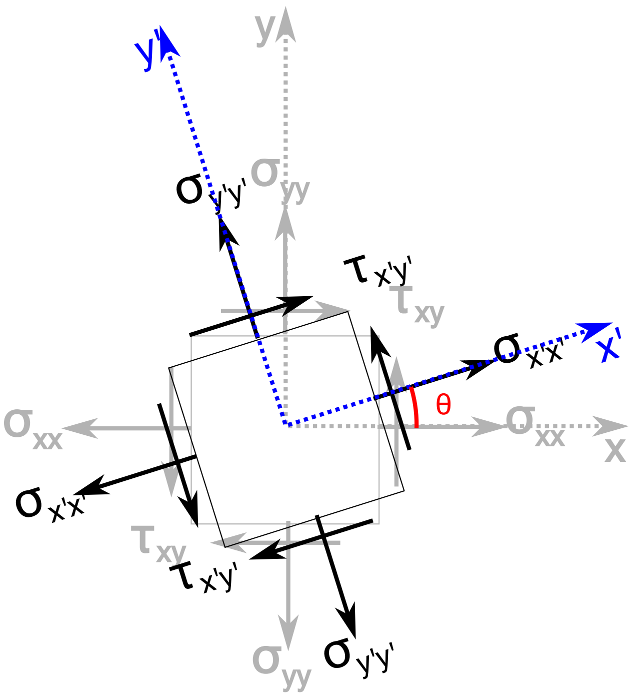
σ_{xx} is 75% of σ_{max}

σ_{yy} is 75% of σ_{max}

τ_{xy} is 75% of τ_{max}

Each of the loads individually do not exceed the fracture limit

But will the plate withstand the combination?



Transformation of stress and strain

We need to find a way to evaluate the stresses and strains in any arbitrary direction.

We already know that we can represent stresses and strains as tensors.

Strictly speaking what we will do is perform a coordinate transform of the tensor to a new coordinate system with the unit vectors in the directions that we are interested in.

We will do matrix coordinate transforms instead

- We can rotate the coordinate system of a vector (x,y) by an angle θ by multiplying with the transformation matrix **Q:**

$$\overrightarrow{v}' = \mathbf{Q} \cdot \overrightarrow{v}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

- We can rotate the coordinate system of a tensor by an angle θ by multiplying with the transformation matrix **Q:**

$$\overleftrightarrow{t}' = \mathbf{Q} \cdot \overleftrightarrow{t}\,\mathbf{Q}^{\mathbf{T}}$$

$$\begin{pmatrix} x'_{11} & x'_{12} \\ x'_{21} & x'_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Transformation of plane stress

- The result of all this are the formulas to transform the normal stress and shear stress from the coordinates x,y to x' and y'

$$\boxed{\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta}$$

$$\boxed{\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta}$$

$$\boxed{\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta}$$

- We can also show from these formulas that

$$\boxed{\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y}$$

Maximum and Minimum Stress

Transformation of stresses and strains

Maximal stresses and principle angles

Transformation of stress and strain

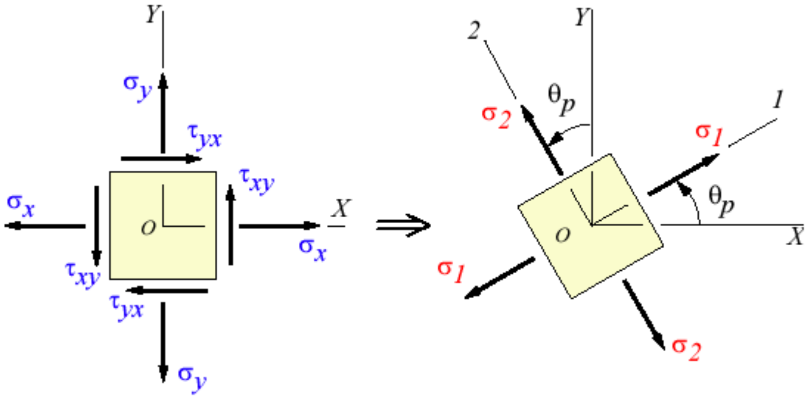
Principal and maximum stresses

- From the equations for normal and shear stress under an arbitrary angle, we can see that there are angles of maximum and minimum shear and normal stresses
- We can calculate these angles by setting the respective derivatives to zero
- For the maximum/minimum of the normal stresses we get:

$$\left(\sigma_{x'}\right)_{\substack{\max\\ \min}} = \sigma_{1\text{ or }2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- This is the *principal stress* and the angle under which it is is the *principal axis*

Transformation to principal stresses



Stresses in given coordinate system

Principal stresses

- Assume an element is under a combination of normal and shear stresses when looked at in a specific coordinate system.
- There exists a rotated coordinate system in which the description of the same stress element will result in only normal stresses, with the shear stresses being zero.
- The normal stresses expressed in this rotated coordinate system are the *principal stresses*. One normal stress is the maximum normal stress. The other normal stress is the minimal stress
- The axes of this rotated coordinate system are the *principal axes*.

Principal and maximum shear stresses

- For the plane where the shear stress is maximum we get:

$$\tau_{\substack{\max\\\min}} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- The absolute value of the maximum shear stress is the same for the axis of maximum and the axis of minimum shear stress. This is understandable, since the material doesn't care if it is "sheared left or right"
- In the principal axis, there is nor shear stress
- In the axis of maximum shear stress, there is also a normal stress (average normal stress)

$$\sigma_{\theta_S} = \frac{\sigma_x + \sigma_y}{2}$$

	Normal Stress	Shear stress
Angle Maximum	$\theta_N = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_{xx}-\sigma_{yy}}\right)$	$\theta_S = \frac{1}{2}\tan^{-1}\left(-\frac{\sigma_{xx}-\sigma_{yy}}{2\tau_{xy}}\right)$
Max/Min Value	$\sigma_{1,2} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$	$\tau_{max,min} = \pm\sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$
“Other” stress at that angle	$\tau_{xy} = 0$	$\sigma_{xy'} = \sigma_{av} = \frac{\sigma_{xx}+\sigma_{yy}}{2}$